# Complete Statistical Analysis of Nonlinear Missile Guidance Systems – SLAM

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**The Statistical Linearization Adjoint Method (SLAM) is a new computerized approach for the complete statistical analysis of nonlinear missile guidance systems through the combination of the CADET method and the adjoint technique. SLAM is an excellent design and analysis tool for missile guidance systems that provides an error budget for the rms miss distance and contains information concerning system behavior.**

## Background

IN designing homing missile guidance systems, a need exists for a computerized method of analysis which relates the effect of statistical disturbances (i.e., measurement noise, random target maneuver, etc.) to the overall performance of the system. The fundamental measure of guidance system performance is generally considered to be the rms miss distance. Assuming a linearized model of the kinematics and guidance system dynamics, two computerized methods are available in order to statistically analyze (generate rms miss distance) a missile guidance system in only one run.

The first method, known as the adjoint technique,[1-3] is based upon the impulse response of the system. This technique not only provides the total rms miss distance† but also shows how each of the disturbance terms contributes to the total rms miss distance. Information is also available concerning the characteristic behavior of the system (i.e., relative stability).

The second method is that of covariance analysis.[1,4,5] In this method, the covariance matrix of the system state vector is propagated as a function of time. Therefore, the rms value of all system states (not only rms miss distance) are available at every instant of time during the flight. The computational requirements of covariance analysis are much greater than the adjoint technique.‡

## Motivation

When important nonlinearities in a stochastic missile guidance system are considered, i.e., acceleration saturation, it becomes necessary to resort to Monte Carlo techniques (repeated simulation trials plus ensemble averaging) to arrive at the rms miss distance. Associated with the Monte Carlo method is the problem that a large number of trials are required to provide confidence in the accuracy of the results.

Recently, an approximate computerized technique known as CADET,[6] was developed for the direct statistical analysis of nonlinear systems. Essentially, this technique employs the statistical linearization[7,8] of nonlinear system elements in conjunction with ordinary covariance analysis to yield statistical performance projections in one computer run. It has been demonstrated[9] that this method, when applied to substantially nonlinear missile guidance systems, yields rms miss distances, with accuracy comparable to that of Monte Carlo runs comprising hundreds of simulated flights.

It seems plausible that since statistical linearization in conjunction with covariance analysis is successful in analyzing nonlinear missile guidance systems, some form of statistical linearization in conjunction with the adjoint method should also be successful. The new features that would be available from a combination of the adjoint method with statistical linearization would be the ability to identify the major contributors to the total rms miss distance and insight into the relative stability of the missile guidance system.

## Review

In this section the adjoint technique, covariance analysis, statistical linearization, and the CADET method are briefly reviewed in order to understand the new method developed in this paper.

### Adjoint Technique

For every linear time-varying system there exists an adjoint system which can be constructed from the original system (given in block diagram form) by application of the following rules§:

1) Replace $t$ by $t_f - t$ in the arguments of all variable coefficients where $t_f$ is the final time.

2) Reverse all signal flow, redefine branch points as sum points, and vice versa. This will make inputs of the original system appear as outputs of the adjoint system and vice versa.

It can be shown that the impulse response of the adjoint system, $h^*$, and the impulse response of the original system, $h$, are related by:

$$h^*(t_f - t_I, t_f - t_0) = h(t_0, t_I) \tag{1}$$

where $t_I$ is the impulse application time and $t_0$ is the observation time. The importance of this relationship manifests itself when it is desired to observe the values of the impulse response function of the original system at the final time, $t_f$, due to various impulse application times. The original system would have to be rerun for each impulse application time in order to generate $h(t_f, t_I)$ as shown in Fig. 1a. However, the adjoint system has to be run only once in order to generate the equivalent impulse response by setting observation time equal

Presented as Paper 77-1094 at the AIAA 1977 Guidance and Control Conference, Hollywood, Fla., Aug. 8-10, 1977; submitted Oct. 21, 1977; revision received May 11, 1978. 

Index categories: Guidance and Control; Simulation; Computational Methods.

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†The adjoint technique can also provide the rms value of any other quantity, such as missile acceleration, at a particular time during the flight.

‡For a continuous $n$ state system, an adjoint solution would require the integration of approximately $n$ equations, whereas a covariance analysis solution would require the integration of $n^2$ equations (approximately $n^2/2$ equations if symmetry is exploited).

§These rules are equivalent to the mathematical definition of an adjoint system when the equations are expressed in state space form.

to the final time in Eq. (1) yielding

$$h^*(t_f - t_I, 0) = h(t_f, t_I) \tag{2}$$

The adjoint impulse response is identical to the impulse response of the original system in every way, except it is generated backwards. The relationship between the two responses is illustrated in Figs. 1a and 1b.

The most important feature of the impulse response of the adjoint system is that it can be used to statistically analyze the original system in the presence of stochastic inputs. The rms response at the terminal time of a linear time-varying system driven by white noise is given by

$$\sigma_{\text{out}}(t_f) = \sqrt{\Phi_{\text{in}} \int_0^{t_f} h^2(t_f, t_I)\,\mathrm{d}t_I} \tag{3}$$

where $\Phi_{\text{in}}$ is the spectral density of the white noise (assumed to be double-sided and stationary)¶ in units of Hz and $\sigma_{\text{out}}$ is the rms value of the output. As discussed previously, the simulation of Eq. (3) is impractical because of the many computer runs needed to generate $h(t_f, t_I)$. However, by invoking Eq. (2) we find that

$$\sigma_{\text{out}}(t_f) = \sqrt{\Phi_{\text{in}} \int_0^{t_f} h^*(t_f - t_I, 0)^2\,\mathrm{d}t_I}$$

$$= \sqrt{-\Phi_{\text{in}} \int_{t_f}^{0} h^*(\tau, 0)^2\,\mathrm{d}\tau} = \sqrt{\Phi_{\text{in}} \int_0^{t_f} h^*(\tau, 0)^2\,\mathrm{d}\tau} \tag{4}$$

Therefore, the rms value of the output of the original system due to a white noise input can be found by squaring, integrating, and then taking the square root of the impulse response of the adjoint system in only one computer run.

The power of the adjoint technique is demonstrated vividly by considering a classical missile intercept problem (linearized guidance loop of a constant velocity interceptor utilizing proportional navigation) as shown in Fig. 2. A triple time-constant guidance system is assumed (one lag for seeker dynamics, one lag for the noise filter, and one lag for autopilot-airframe dynamics). The target is assumed to have constant velocity and its lateral acceleration is modeled as the output of a low-pass filter driven by white noise. This model has the same autocorrelation function as a jinking maneuver (constant magnitude maneuver with random sign switching). It is assumed that the seeker measurement of the line-of-sight angle rate is corrupted by white glint noise and white fading (range independent) noise with spectral densities $\Phi_{sn}$ and $\Phi_{fn}$, respectively. The miss distance, $y(t_f)$, is defined as the relative missile-to-target displacement at the point of closest approach. For simplicity, the "head-on" intercept case is considered. The nominal values of all system parameters used are tabulated in Table 1.

Using the two rules given at the beginning of this section, an adjoint model of the kinematic guidance loop was constructed as is shown in Fig. 3. According to theory, an impulse should be applied to the adjoint system at the equivalent location ($\dot{x}4$) to where the quantity of interest in the original system is output ($y$). For simulation purposes, an initial condition of unity on $x4$ rather than a unit impulse on its derivative is used. Note that the three inputs to the original system (target maneuver, glint noise, and fading noise) become outputs in the adjoint system (miss sensitivities due to target maneuver, glint noise, and fading noise). Since the sensitivity coefficients of the adjoint system do not depend on the spectral density levels of the error sources, the program does not have to be rerun if the spectral density levels change. Superposition permits the calculation of the total rms miss distance to be:

$$\sigma_y(t_F) = \left\{ \left[ \frac{\sigma_y^2(t_F)\,|\,\text{Tgt Mvr}}{\Phi_s} \right] \cdot \Phi_s + \left[ \frac{\sigma_y^2(t_F)\,|\,\text{Glint}}{\Phi_{sn}} \right] \cdot \Phi_{sn} + \left[ \frac{\sigma_y^2(t_F)\,|\,\text{Fading}}{\Phi_{fn}} \right] \cdot \Phi_{fn} \right\}^{1/2} \tag{5}$$

The rms level of the individual miss distance contributors along with the total rms miss distance are plotted vs adjoint time in Fig. 4. (Adjoint time can be interpreted here as either time-of-flight or time-to-go at which disturbances occur.) Note that for this system the major contributor to the miss distance is glint noise. At minimal extra expense, other disturbance sensitivities can also be obtained in order to further quantify system behavior. For example, in Fig. 5, the

¶It is important to note that no significant loss of generality is suffered by restricting the analysis to systems with a stationary white noise input, because in theory the weighting function of the required shaping filter can be included in the weighting function of the actual system.

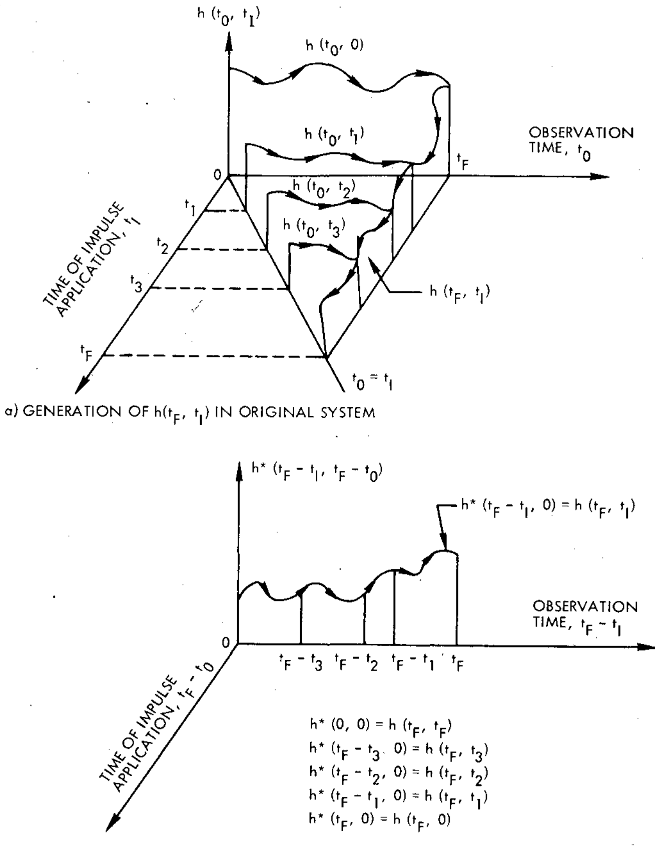


**Fig. 1 Impulse responses of original and adjoint systems are related: a) generation of $h(t_F, t_I)$ in original system, b) generation of $(t_F, t_I)$ in adjoint system.**

**Table 1 Nominal values of all system parameters**

| Nominal conditions | Nominal value specifications |
|---|---|
| Seeker bandwidth, $\omega_1$ | 20 rad/s |
| Noise filter bandwidth, $\omega_2$ | 10 rad/s |
| Autopilot-airframe bandwidth, $\omega_3$ | 10 rad/s |
| Target maneuver bandwidth, $2\nu$ | $0.2\ \text{s}^{-1}$ |
| rms target acceleration, $B$ | 161 ft/s$^2$ |
| Closing velocity, $Vc$ | 3000 ft/s |
| Effective navigation ratio, $N'$ | 3 |
| Spectral density of glint noise, $\Phi_{sn}$ | 4 ft$^2$/Hz |
| Spectral density of fading noise, $\Phi_{fn}$ | $1 \times 10^{-6}$ rad$^2$/Hz |
| Time of flight, $t_F$ | 5 s |

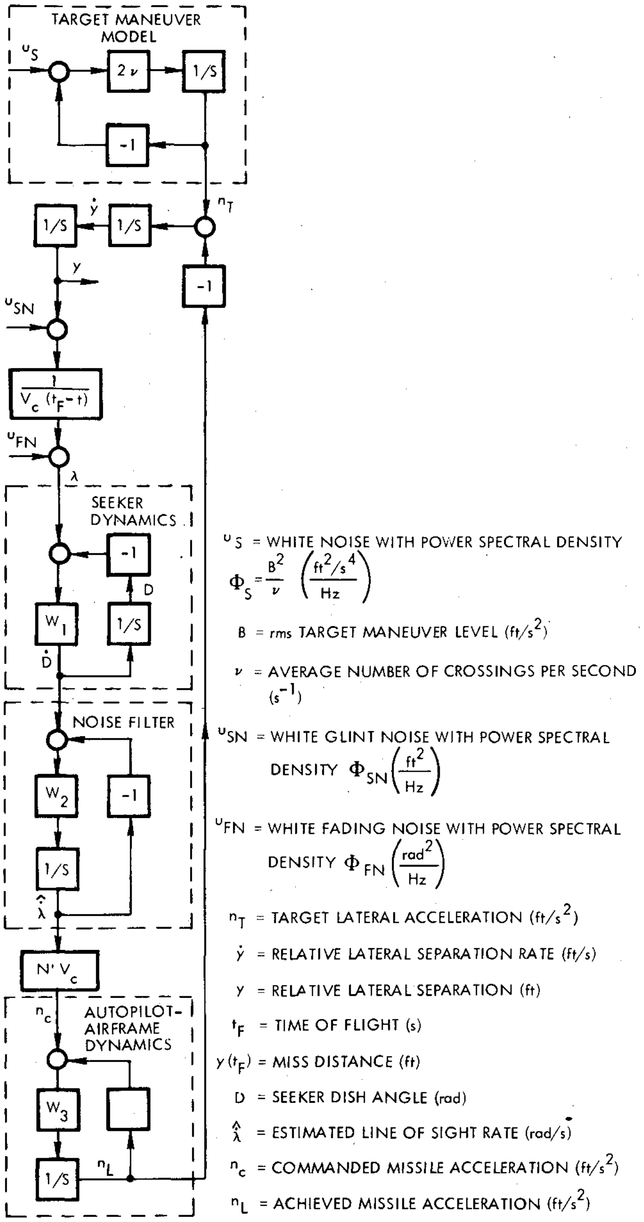


**Fig. 2 Linear kinemetic guidance loop (original system).**

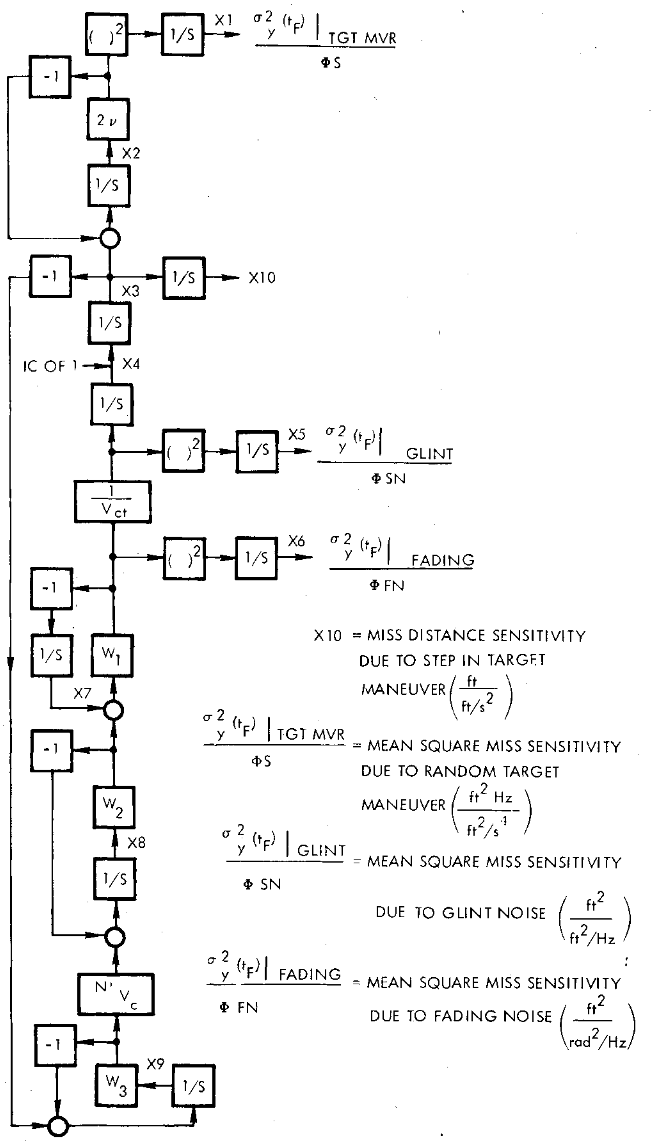


**Fig. 3 Linear kinematic guidance loop (adjoint system).**

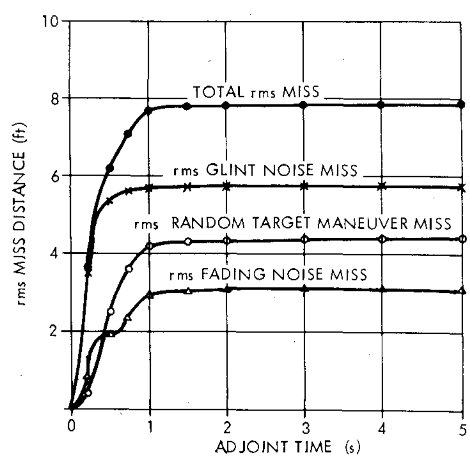


**Fig. 4 rms miss distance error budget.**

miss distance sensitivity due to a step in target acceleration ($x10$) is plotted against adjoint time. Figure 5 indicates that the optimal time for the target to maneuver (to maximize miss distance) is about 0.6 s before intercept. It is also apparent from this curve that if the target maneuvers too soon (adjoint time large), the resulting miss distance will be small. In a well-designed missile guidance system, the miss distance sensitivity curve for a step in target acceleration always approaches zero as adjoint time approaches infinity. The amount of adjoint time it takes for this curve to settle down is directly related to the overall guidance system time constant. Therefore, it can be seen that a great deal of information concerning system performance and behavior is available from one adjoint solution.

**Covariance Analysis**

The dynamics of a linear time-varying stochastic system can be represented by the following first-order vector differential equation

$$\dot{x}=F(t)x(t)+u(t) \tag{6}$$

where $x(t)$ is the system state vector and $u(t)$ is a white noise vector with spectral density matrix, $Q(t)$. The differential equation for the propagation of the covariances is[4,5]

$$\dot{X}=F(t)\cdot X+X\cdot F^T(t)+Q(t) \tag{7}$$

The diagonal elements of $X(t)$ are the mean square values of the state variables** and the off-diagonal elements represent

**For simplicity, all random quantities are assumed to have zero mean in this paper.

the degree of correlation between the various state variables. Therefore, the integration of Eq. (7) represents another direct method of analyzing the statical properties of $x(t)$ in one computer run.

The usefulness of covariance analysis is easily demonstrated by again considering the linear kinematic guidance loop of Fig. 2. The system equation in matrix form becomes

$$\begin{bmatrix} \dot{n}_T \\ \ddot{y} \\ \dot{y} \\ \dot{D} \\ \dot{\hat{\lambda}} \\ \dot{n}_L \end{bmatrix} = \begin{bmatrix} -2\nu & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\omega_I}{Vc(t_F-t)} & -\omega_I & 0 & 0 \\ 0 & 0 & \dfrac{\omega_I\omega_2}{Vc(t_F-T)} & -\omega_I\omega_2 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & N^I Vc\omega_3 & -\omega_3 \end{bmatrix} \cdot \begin{bmatrix} n_T \\ \dot{y} \\ y \\ D \\ \hat{\lambda} \\ n_L \end{bmatrix} + \begin{bmatrix} 2\nu u_s \\ 0 \\ 0 \\ \omega_I\left(u_{fn} + \dfrac{u_{sn}}{Vctgo}\right) \\ \omega_I\omega_2\left(u_{fn} + \dfrac{u_{sn}}{Vctgo}\right) \\ 0 \end{bmatrix} \tag{8}$$

$$\dot{x} = F \cdot x + u$$

Equation (7) is then integrated to find $X(t)$ where $F(t)$ is defined in Eq. (8) and $Q(t)$, which can be found from Eq. (8), is given by

$$Q(t) = \begin{bmatrix} 4\nu B^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_I^2\left[\Phi_{fn} + \dfrac{\Phi_{sn}}{Vc^2(t_F-t)^2}\right] & \omega_I^2\omega_2\left[\Phi_{fn} + \dfrac{\Phi_{sn}}{Vc^2(t_F-t)^2}\right] & 0 \\ 0 & 0 & 0 & \omega_I^2\omega_2\left[\Phi_{fn} + \dfrac{\Phi_{sn}}{Vc^2(t_F-t)^2}\right] & \omega_I^2\omega_2^2\left[\Phi_{fn} + \dfrac{\Phi_{sn}}{Vc^2(t_F-t)^2}\right] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

Figure 6, which was obtained by simulating Eq. (7), presents a plot of the rms relative separation between the missile and target, $\sqrt{X(3,3)}$, as a function of time. At the end of flight, the value of this quantity is the rms miss distance

$$\text{rms miss distance} \triangleq \sqrt{X(3,3)}\,|_{t=t_F} \tag{10}$$

It can be seen that the values of the miss distances resulting from the adjoint technique and covariance analysis are identical. In covariance analysis, statistical information concerning all the states, such as rms acceleration, shown in Fig. 7 are also available so that it is also possible to validate the assumption concerning system linearity (i.e., no acceleration saturation).

#### Statistical Linearization

A useful tool in analyzing nonlinear systems having random inputs is the method of statistical linearization. With this technique, the nonlinear element is replaced by an equivalent gain, where the gain depends upon the assumed form of the input signal to the nonlinearity. Booton[7] developed a simple technique, which will be shown later, for finding the equivalent gain. Consider a nonlinear system with input $x(t)$ and output $y(t)$, in which we would like to replace the nonlinear element with some equivalent gain $K_{eq}$. The error signal, $e(t)$, is defined as the difference between $y(t)$ and the equivalent gain output. For purposes of this paper, let us assume that $x(t)$ is a zero-mean random process. We can find $K_{eq}$ by first computing the mean-square value of the error signal $e(t)$.

$$\overline{e} = \overline{y^2} - 2K_{eq}\overline{xy} + K_{eq}{}^2\overline{x^2} \tag{11}$$

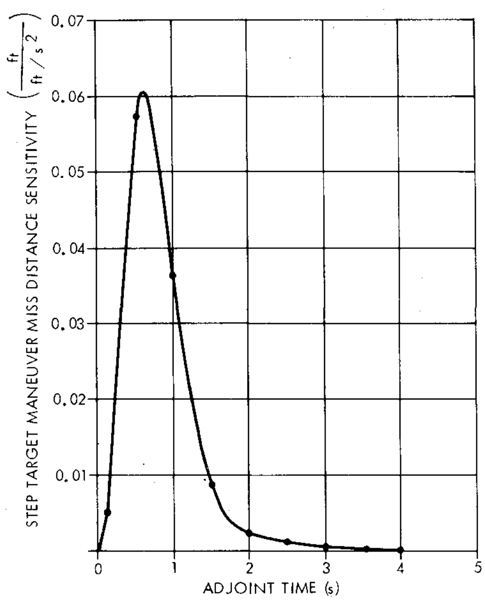


**Fig. 5 Miss distance sensitivity due to a step in target maneuver vs adjoint time.**

We can find the minimum value of $\overline{e}^2$ by setting its derivative equal to zero yielding

$$K_{eq} = \overline{xy}/\overline{x^2} = -\int_{-\infty}^{\infty} xyp(x)\,\mathrm{d}x \Big/ \int_{-\infty}^{\infty} x^2 p(x)\,\mathrm{d}x \tag{12}$$

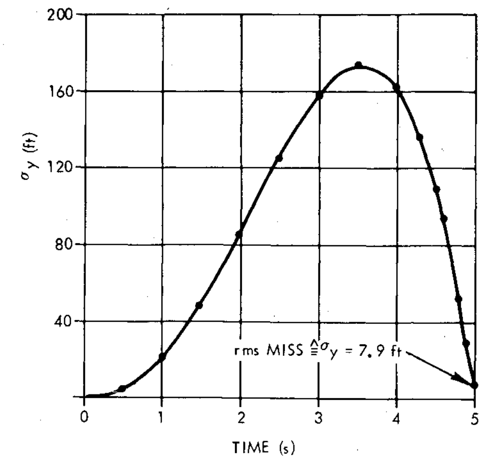


**Fig. 6 rms relative missile – target separation vs time.**

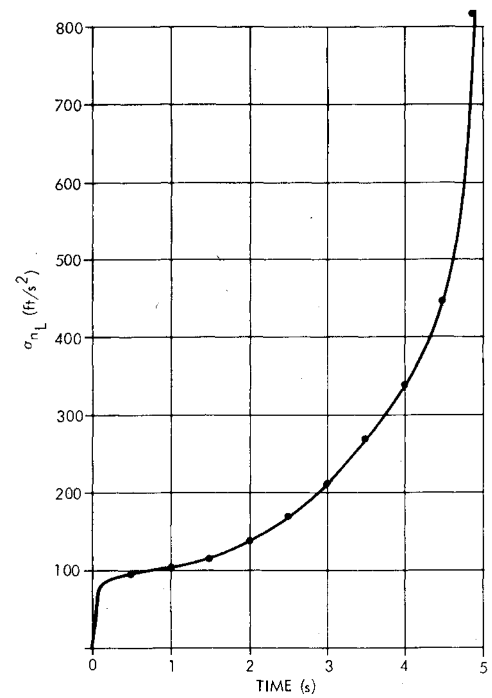


**Fig. 7 rms achieved acceleration vs time.**

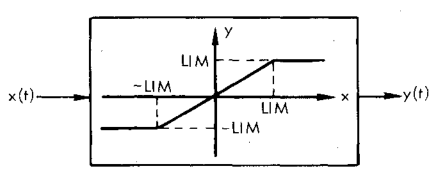


**Fig. 8 Input-output characteristics of a limiter.**

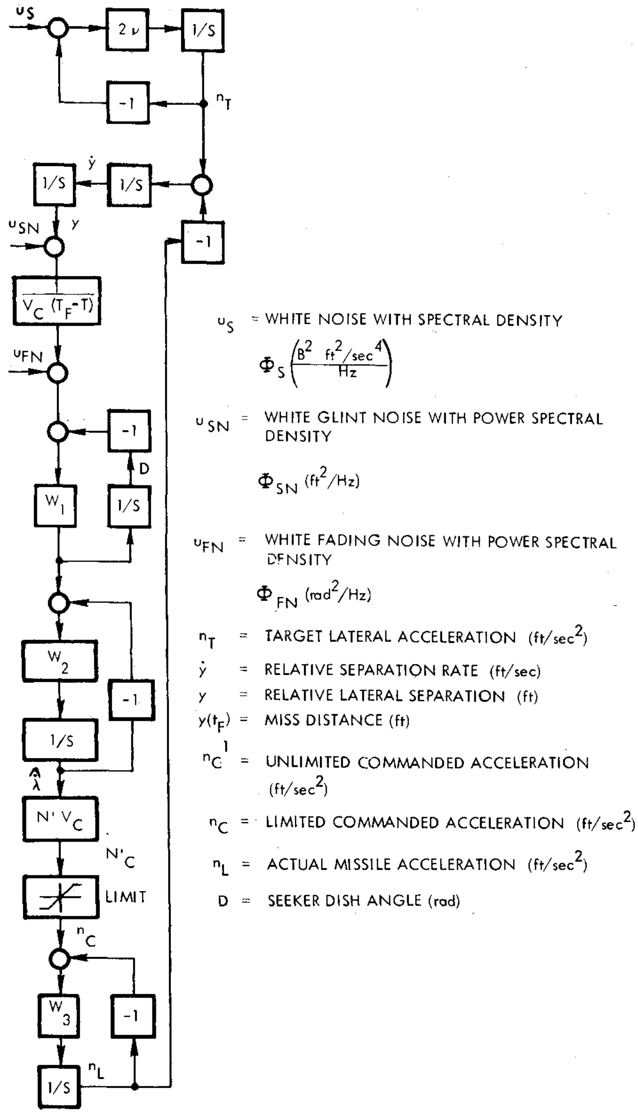


**Fig. 9 Nonlinear kinematic guidance loop.**

If we assume that the input signal $x(t)$ is a zero-mean Gaussian random process with the following probability density function,

$$p(x) = (1/\sigma_x\sqrt{2\Pi})e^{-x^2/2\sigma_x^2} \tag{13}$$

where $\sigma_x$ is the rms value of $x(t)$, then the equivalent gain becomes

$$K_{eq} = \frac{1}{\sigma_x^3\sqrt{2\Pi}}\int_{-\infty}^{\infty} xye^{-x^2/2\sigma_x^2}dx \tag{14}$$

Input-sensitive gains of this type, which approximate the transfer characteristics of the nonlinearity, are termed random input describing functions and are tabulated for the most important nonlinearities in Ref. 8. Statistical linearization may be demonstrated by calculating the describing function for one of the most important nonlinearities in a missile guidance system, namely the acceleration saturation. Consider the limiter of Fig. 8.

The describing function, as calculated from Eq. (14), becomes:

$$K_{eq} = \frac{-\lim}{\sigma_x^3\sqrt{2\Pi}}\int_{-\infty}^{-\lim} xe^{-x^2/2\sigma_x^2}dx + \frac{1}{\sigma_x^3\sqrt{2\Pi}}\int_{-\lim}^{\lim} x^2e^{-x^2/2\sigma_x^2}dx + \frac{\lim}{\sigma_x^3\sqrt{2\Pi}}\int_{\lim}^{\infty} xe^{-x^2/2\sigma_x^2}dx \tag{15}$$

Evaluation of the preceding integral leads to:

$$K_{eq} = \frac{1}{\sigma_x\sqrt{2\Pi}}\int_{-\lim}^{\lim} e^{-x^2/2\sigma_x^2}dx \tag{16}$$

Equation (16) can be rewritten in terms of the probability integral as:

$$K_{eq} = 2\left[\frac{1}{\sigma_x\sqrt{2\Pi}}\int_{-\infty}^{\lim} e^{-x^2/2\sigma_x^2}dx\right] - 1 \tag{17}$$

The preceding integral can be found by table lookup or can be approximated to five-place accuracy[10] by:

$$K_{eq} = 1 - (2/\sqrt{2\Pi})e^{-\lim^2/2\sigma_x^2} \times [0.4361836\omega - 0.1201676\omega^2 + 0.937298\omega^3] \quad (18)$$

where

$$\omega = \frac{1}{1 + [0.33267 \times \lim]/\sigma_x} \quad (19)$$

The describing function for the limiter depends, as one would expect, only on the value of the limit lim and the rms value of the input signal $\sigma_x$.

**CADET Method**

The Covariance Analysis Describing Function Technique (CADET) is an approximate computerized technique for analyzing the statistical behavior of nonlinear stochastic systems. The principal steps to be followed in the application of the CADET method to missile guidance systems are:

1) Replace each nonlinear element by its corresponding random input describing function gain, based upon an assumed Gaussian probability density function†† for the input to the nonlinearity.

2) Using the resulting linear system model, employ conventional covariance analysis techniques to propagate the statistics of the system state vector, recognizing that the describing function gains are functions of those statistics.

3) Compute the rms miss distance at the intercept time from the elements of the system covariance matrix.

The CADET method may be demonstrated by once again considering the example of Fig. 2, only this time with an acceleration limit on the commanded acceleration as shown in Fig. 9. The acceleration saturation nonlinearity is first replaced by a random input describing function, $K_{\lim}$. The linearized system equation then becomes

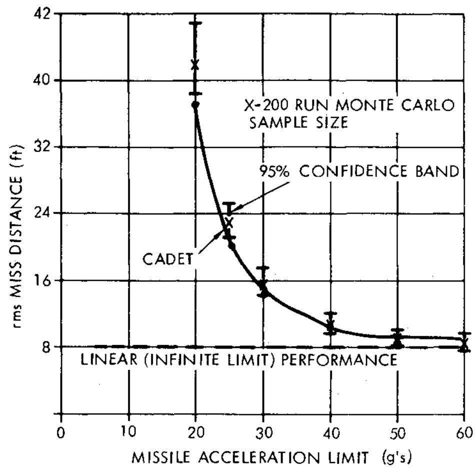


**Fig. 10 Typical CADET performance analysis.**

and the results of this study are shown in Fig. 10. Only six CADET runs were needed to generate these results. Superimposed on these figures are similar results generated using the Monte Carlo method with a sample size consisting of 200 runs. The 95% confidence intervals (assuming a gaussian distribution of the miss distance statistics) for the Monte Carlo results are also indicated in the figure. It can be seen from this example that CADET is extremely accurate. Reference 9 also obtains results indicating CADET accuracy to be equivalent to Monte Carlo sets consisting of hundreds of flights. Figure 10 shows that the missile acceleration limit has a profound influence on the rms miss distance. The dashed curve shows the results of a linear system analysis (infinite acceleration limit) obtained from either Fig. 4 or 6. It can be seen that for this case at least six or seven times the acceleration capability of the target (30-35 $g$ capability needed against 5 $g$ target) is needed by the missile in order to keep the rms miss distance close to the linear level (near knee of curve).

$$\begin{bmatrix} \dot{n}_T \\ \ddot{y} \\ \dot{y} \\ \dot{D} \\ \dot{\hat{\lambda}} \\ \dot{n}_L \end{bmatrix} = \begin{bmatrix} -2\nu & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\omega_1}{Vc(t_F-t)} & -\omega_1 & 0 & 0 \\ 0 & 0 & \frac{\omega_1\omega_2}{Vc(t_F-t)} & -\omega_1\omega_2 & -\omega_2 & 0 \\ 0 & 0 & 0 & 0 & N'VcK_{\lim}\omega_3 & -\omega_3 \end{bmatrix} \cdot \begin{bmatrix} n_T \\ \dot{y} \\ y \\ D \\ \hat{\lambda} \\ n_L \end{bmatrix} + \begin{bmatrix} 2\nu u_s \\ 0 \\ 0 \\ \omega_1\left[U_{fn} + \frac{U_{sn}}{Vc(t_F-t)}\right] \\ \omega_1\omega_2\left[U_{fn} + \frac{U_{sn}}{Vc(t_F-t)}\right] \\ 0 \end{bmatrix} \quad (20)$$

Equation (7) is then integrated to find $X(t)$ where $Q$ is still given by Eq. (9) and $F$ is obtained from Eq. (20).

The describing function gain for the limiter (derived in the previous section) is a function of the statistics of the unlimited commanded acceleration, $n_c'$ and the limit level, $n_{\lim}$, and can be computed from Eqs. (18) and (19). The rms level of the input signal to the nonlinearity $\sigma_{nc}'$ is calculated by first expressing the unlimited commanded acceleration $n_c'$ as a function of the states. The mean-square value then becomes

$$\sigma^2_{n_c'} = (N'V_c)^2 X(5,5) \quad (21)$$

A CADET program was constructed for the system of Fig. 9 using the input values of Table 1. Cases were run in which the missile acceleration limit, $n_{\lim}$, was treated as a parameter

††Although at first this assumption appears to be very restrictive, it is not because most dynamical systems contain more linear than nonlinear elements. The low-pass filtering in these systems insures that nongaussian nonlinearity outputs result in nearly gaussian inputs, as signals circulate in the system of interest.

## Methodology

An exceptionally promising technique developed specifically for the complete statistical analysis of nonlinear homing missile guidance systems is now presented. It is called the Statistical Linearization Adjoint Method (SLAM). Basically this technique uses the CADET method in conjunction with the adjoint technique. The principal steps in SLAM are:

1) Replace each nonlinear element in the original system by its corresponding random input describing function gain, based upon an assumed Gaussian probability density function for the input to the nonlinearity.

2) Using the resulting linear system model, employ conventional covariance analysis techniques to propagate the statistics of the system state vector.

3) Store the resulting describing function gains for each nonlinearity as a function of time.

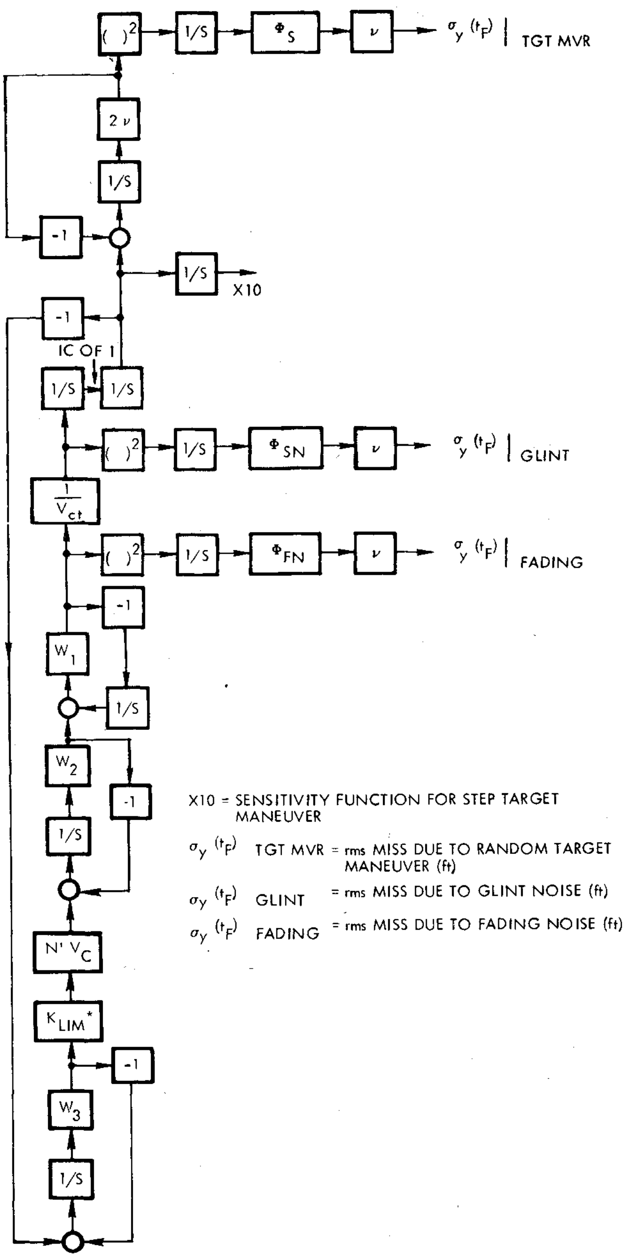


**Fig. 11 SLAM model of nonlinear kinematic guidance loop.**

4) Generate an adjoint model of the linearized system model by replacing $t$ by $t_F - t$ in the arguments of all variable coefficients (including describing function gains) and reversing signal flow ‡‡ so that the inputs of the original system will appear as outputs of the adjoint system.

The SLAM system will not only yield the identical rms miss distance to that of the CADET system, but will also show how the individual error disturbances contribute to the total rms miss distance. This error budget is approximate and, strictly speaking, is valid only for the case for which the $K_{\text{lim}}{}^*$ time history is valid. Therefore, it is not safe to extrapolate the results to obtain miss distances for different error source input levels. However, this type of error disturbance breakdown is extremely useful in that it flags the major contributors to the total rms miss distance in the nonlinear system. This error budget is no less accurate but a lot less

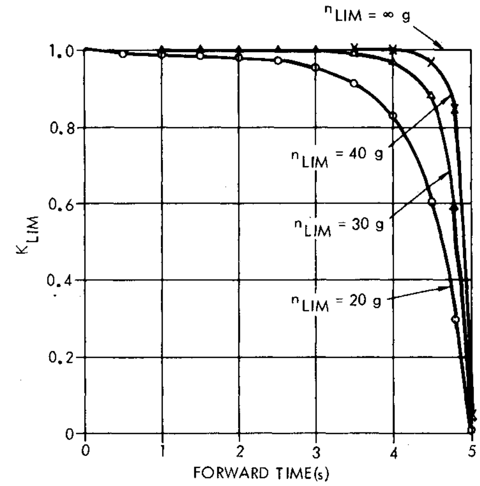


**Fig. 12 Describing functions for various acceleration limits vs forward time.**

expensive than one generated by the brute force method§§ when Monte Carlo or CADET techniques are used.

Sensitivity functions (i.e., sensitivity due to a step-target maneuver), can also be printed out at no extra cost in order to get an indication of system behavior, i.e., relative stability. Again strictly speaking, these sensitivity functions have no meaning in the sense that miss distances can be calculated from them. However, since these sensitivity functions, such as the sensitivity due to a step target maneuver, represent the impulse response of the system, valuable insight into system behavior can be gained by monitoring this output.

The SLAM concept is also useful in that it has a self-checking capability. That is, if the rms miss distance of the adjoint portion of the program does not agree with the rms miss distance of the CADET portion, it is known that either a programming or conceptual error exists within the program. Of course no system is perfect and with ingenuity it is still possible for the user to make undetected errors, but it is felt that the SLAM concept considerably reduces this possibility.

Computationally, SLAM is far more efficient than the Monte Carlo technique and only slightly more burdensome computationally than the CADET technique. This is true since the implementation of SLAM requires the integration of only an extra $n$ equations (CADET already requires the integration of $n^2$ or $n^2/2$ equations depending whether or not symmetry is exploited). In systems with several nonlinearities, the storage requirements imposed by SLAM are minor. In systems with many nonlinearities and/or long trajectories, any storage problems that may arise can be alleviated by taking (in the adjoint portion of SLAM) less samples of the describing functions. This approximation will only change the final results slightly.

## Illustrative Example of SLAM

As a demonstration of the utility of SLAM, the nonlinear stochastic guidance system of Fig. 9 is reconsidered. A CADET portion of the SLAM program is first generated using the inputs of Table 1. The CADET portion of the program is run in order to find the time history of the describing function, $K_{\text{lim}}$. The SLAM program then reverses, in time, this describing function (replace $t$ by $t_F - t$) yielding $K_{\text{lim}}{}^*$. The reversed describing function gain, $K_{\text{lim}}{}^*$, is then entered into a linearized adjoint model of the original

‡‡It is interesting to note that the adjoint equations could have been developed directly from the state equations used in the CADET portion of the program. However to do so would eliminate any possibility of giving SLAM a self-checking feature.

§§Monte Carlo or CADET results must be generated with only one error source at a time. The total rms miss distance can then be calculated by appropriately combining the miss contributions from each error source. In nonlinear systems, the total rms miss (obtained from running all error inputs at once) does not necessarily equal the appropriate combination of the individual error sources.

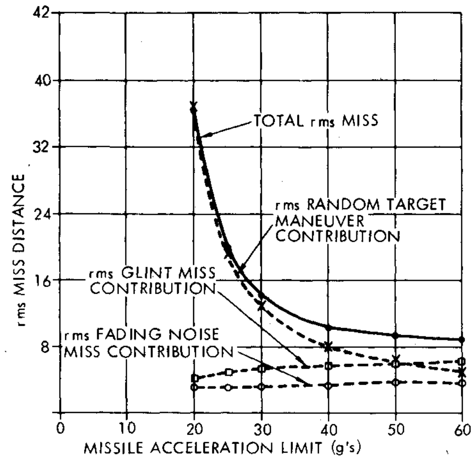


**Fig. 13 Typical SLAM performance analysis.**

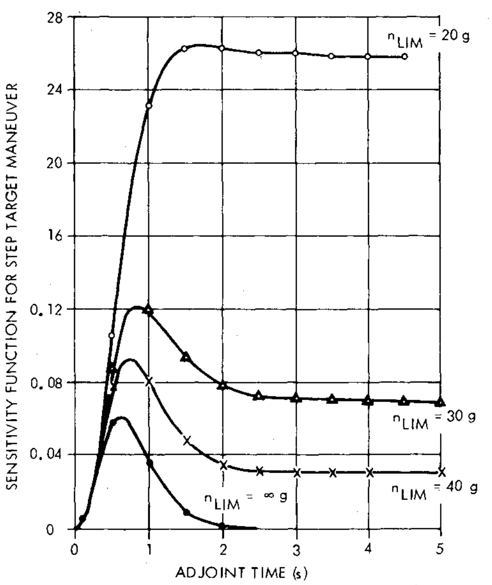


**Fig. 14 Sensitivity function for step target maneuver for various acceleration limits.**

nonlinear system, as shown in Fig. 11. The adjoint portion of the SLAM program is then run in order to generate an approximate error budget for the nonlinear system.

A program utilizing the SLAM technique was run with the acceleration limit, $n_{\lim}$, as a parameter. The describing function gains, resulting from the CADET portion of the program, are shown for typical values of $n_{\lim}$ in Fig. 12. As theory predicts, the gains are unity when no saturation is taking place, and they decrease in value as the saturation becomes deeper.

The total rms miss distance error budget obtained from SLAM as a function of the missile acceleration limit is plotted in Fig. 13. Note that the total rms miss distance for each of the acceleration limits corresponds exactly to the results of the CADET program (see Fig. 10). This must be true if no programming or conceptual errors have been made in implementing SLAM. The error budget tells us that when the system is only slightly in saturation ($n_{\lim} = 60\ g$), the major contributor to the total rms miss distance is glint noise (see Fig. 4 for confirmation). As saturation increases ($n_{\lim}$ decreases), the major contributor to the total rms miss distance quickly becomes the random target maneuver disturbance. On the other hand, the contributions to the miss from fading and glint noise slightly decrease as saturation increases. In this case, saturation acts as additional filtering and thus the result is not surprising. This type of error disturbance breakdown provided by SLAM can be extremely useful to the guidance system designer in developing a balanced system.

Finally, Fig. 14 is a plot of the sensitivity function due to a step-target maneuver. Although this sensitivity function cannot be used in calculating the miss distance due to a step-target maneuvers,¶ it can be used in gaining insight into system behavior. Mathematically, it is useful because it represents the impulse response of the quasilinearized system. (Note from Fig. 11 that the derivative of this function is used in calculating the miss distance due to a random maneuver.) Figure 14 shows that in the linear case ($n_{\lim} = \infty$), the curve peaks and then quickly approaches zero (see also Fig. 5). This should be the case in a well-designed missile guidance system employing proportional navigation. That is, a target maneuver occurring at long times-to-go (at least ten guidance time constants) should not contribute to the miss distance. However, as can be seen from Fig. 14, the introduction of an acceleration limit causes the miss distance sensitivity to approach asymptotic values other than zero (this is common in proportional navigation systems which are using too small a value for the effective navigation ratio). For saturation even greater than shown in Fig. 14, the sensitivity function would increase monotonically. Although, strictly speaking, this is not an unstable system, it is a system which cannot guide effectively on maneuvering targets. This type of information is also valuable to the guidance system designer.

## Conclusions

SLAM is a powerful new computerized approach to the complete statistical analysis of nonlinear missile guidance systems. With this new technique, an approximate error budget is automatically generated indicating how each of the disturbances contributes to the total rms miss distance. At no extra cost, sensitivity f7nctions, such as the one due to a step in target maneuver, can be obtained so that relative system stability can be investigated. Finally, the SLAM concept is useful in that it has a self-checking capability. That is, if the rms miss distance of the adjoint portion of the program does not agree with the rms miss distance of the CADET portion, it is known that either a programming or conceptual error exists within the program. Although the theory discussed in this paper applies to continuous systems, SLAM can easily be extended to discrete and mixed continuous-discrete systems.

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¶This is due to the fact that the reversed describing function gains, $K_{\lim}$, are valid only for one time-of-flight and one set of random disturbance values.